Hecke operators

Silverman "Advanced topics in..."
Chapter 1 : Sections 9–11

Modularity condition

$$\left\{ f : \mathcal{H} \to \mathbb{C}, \qquad f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \right.$$

in terms of lattices:

$$(*)\ F : \{\text{lattices } \Lambda \subset \mathbb{C}\} \longrightarrow \mathbb{C}, \qquad F(\lambda\Lambda) = \lambda^{-k} F(\Lambda) \quad \forall \lambda \in \mathbb{C}^\times$$

$$f(z) = F(\mathbb{Z}z + \mathbb{Z}), \qquad F(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2) = \omega_2^{-k} f\left(\frac{\omega_1}{\omega_2}\right)$$

Given $m \in \mathbb{N}$, F as above, define

$$(T_m F)(\Lambda) = \sum_{\Lambda' \subset \Lambda,\ [\Lambda : \Lambda'] = m} F(\Lambda')$$

This induces an operator $T_m : M_k(\Gamma_1) \longrightarrow M_k(\Gamma_1)$ given by

$$(T_m f)(z) = m^{k-1} \sum_{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_1 \backslash \mathcal{D}_m} \boxed{(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right)}$$

$\Gamma_1 = SL_2(\mathbb{Z})$

$$\mathcal{D}_m = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{Z}) \,\middle|\, ad-bc=m \right\}$$

[system of representatives $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ with $ad=m$, $0 \leq b < d$]

$$= m^{k-1} \sum_{\substack{ad=m \\ a>0 \\ d>0}} \frac{1}{d^k} \sum_{0 \leq b < d} f\left(\frac{az+b}{d}\right)$$

If f has Fourier expansion

$$f(q) = \sum_{n=0}^{\infty} a_n(f) q^n$$

$$z \longmapsto q = e^{2\pi i z}$$

$$T_m : M_k(\Gamma_1) \to M_k(\Gamma_1)$$

then

$$(T_m f)(q) = \sum_{d \mid m} \left(\frac{m}{d}\right)^{k-1} \sum_{n=0}^{\infty} a_n(f) q^{mn/d^2}$$

$$= \sum_{n=0}^{\infty} \underbrace{\left(\sum_{r \mid \gcd(m,n)} r^{k-1} a_{mn/r^2}(f) \right)}_{a_n(T_m f)} q^n$$

$$a_0(T_m f) = \sigma_{k-1}(m)\, a_0(f)$$

Observations:

- get $T_m : S_k(\Gamma_1) \to S_k(\Gamma_1)$
- T_m preserves integrality (the ring of coefficients of f)
- $\{T_m \mid m \in \mathbb{N}\}$ commute.

Examples

$S_{12}(\Gamma_1) = \mathbb{C} \cdot \Delta$ so $T_m \Delta =$ multiple of Δ

$\Delta(q) = q - 24q^2 + \dots$

$a_1(T_m f) = a_m(f)$

$(T_m \Delta)(q) = a_m(\Delta) q + \dots = \tau(m) q + \dots$

$\Rightarrow T_m \Delta = \tau(m) \Delta \Rightarrow a_n(T_m \Delta) = \tau(m)\tau(n)$

$$a_n(T_m\Delta) = \tau(m)\tau(n)$$
$$\qquad \| $$
$$\sum_{r \mid \gcd(m,n)} r^{11}\, \tau\left(\frac{mn}{r^2}\right)$$

In particular, if $\gcd(m,n)=1$ then
$$\tau(mn) = \tau(m)\tau(n)$$
(conjectured by Ramanujan, proved by Hecke)

Generalisation: if $f \in M_k(\Gamma_1)$ is a simultaneous eigenvector for all T_m $(m \in \mathbb{N})$ with eigenvalue λ_m, then
$$a_m(f) = \lambda_m a_1(f) \qquad \forall m \in \mathbb{N}$$
So $a_1(f) \neq 0$, can normalise so that $a_1(f)=1$.

(get a <u>normalised Hecke eigenform</u>).

Then $\quad T_m f = a_m(f)\, f \qquad \lambda_m = a_m(f)$

$$a_m(f)\, a_n(f) = \sum_{r \mid \gcd(m,n)} r^{k-1}\, a_{\frac{mn}{r^2}}(f)$$

For G_k, $k \geq 4$ we have $\quad T_m G_k = \sigma_{k-1}(m)\, G_k$

For $k = 16, 18, 20, 22, 26$, have $\dim S_k(\Gamma_1) = 1$

have $\Delta E_{k-12} \in S_k(\Gamma_1)$.

$k = 24$: $\dim S_{24}(\Gamma_1) = 2$. $\quad$ Basis $\{\Delta E_4^3, \Delta^2\}$.

Hecke proved that there is a unique basis of normalised Hecke eigenforms for $M_k(\Gamma_1)$.

$\Gamma_1 = SL_2(\mathbb{Z})$

$\Gamma_0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_1 \;\middle|\; \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \bmod N \right\}$

L-series

f normalised Hecke eigenform

$$\begin{cases} a_m a_n = a_{mn} & \text{if } \gcd(m,n)=1 \\ a_{p^\nu} a_p = a_{p^{\nu+1}} + p^{k-1} a_{p^{\nu-1}} & \text{if } p \text{ prime}, \nu \geq 1. \end{cases}$$

$$L(f,s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_{p \text{ prime}} \left(1 + \frac{a_p}{p^s} + \frac{a_{p^2}}{p^{2s}} + \cdots \right)$$

$$= \prod_{p \text{ prime}} \frac{1}{1 - a_p p^{-s} + p^{k-1-2s}}$$

Euler product expansion of $L(f,s)$.